Mixed Integer Linear Programming

Part III - exercises

Prof. Davide M. Raimondo

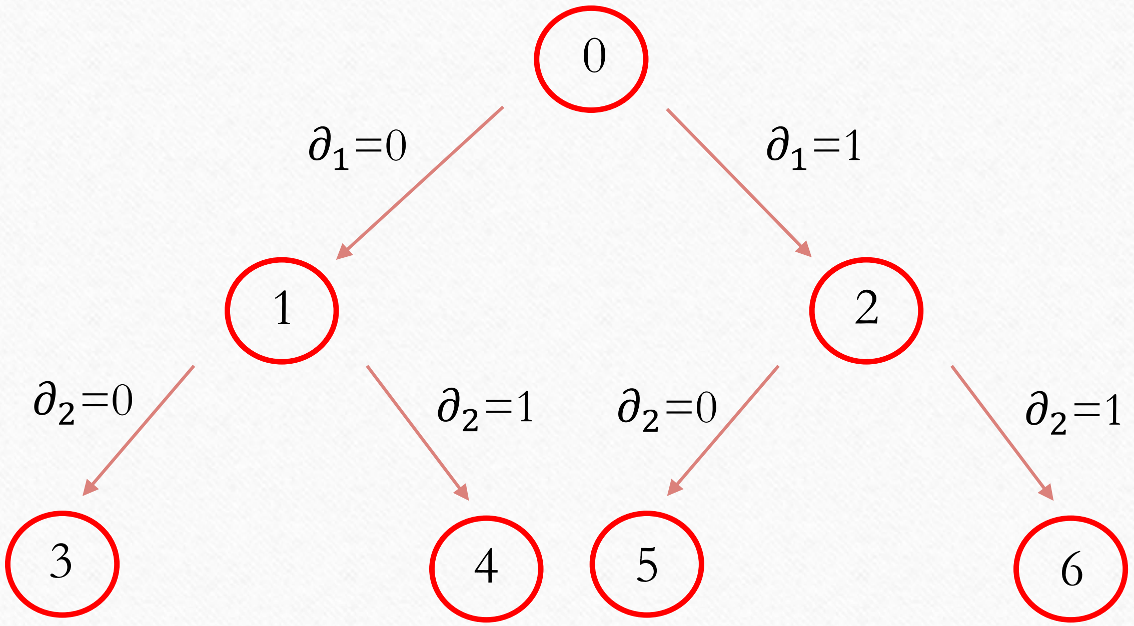
Exercise 1

$$\begin{aligned} \min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\ & x_1 + 10x_2 + 2y_1 + y_2 \geq 100 \\ & y_1 + y_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

$\xrightarrow{\text{standard form}}$

$$\begin{aligned} \min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\ & x_1 + 10x_2 + 2y_1 + y_2 - s_1 = 100 \\ & y_1 + y_2 + s_2 = 1 \\ & x_1, x_2, s_1, s_2 \geq 0 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

Exercise 1

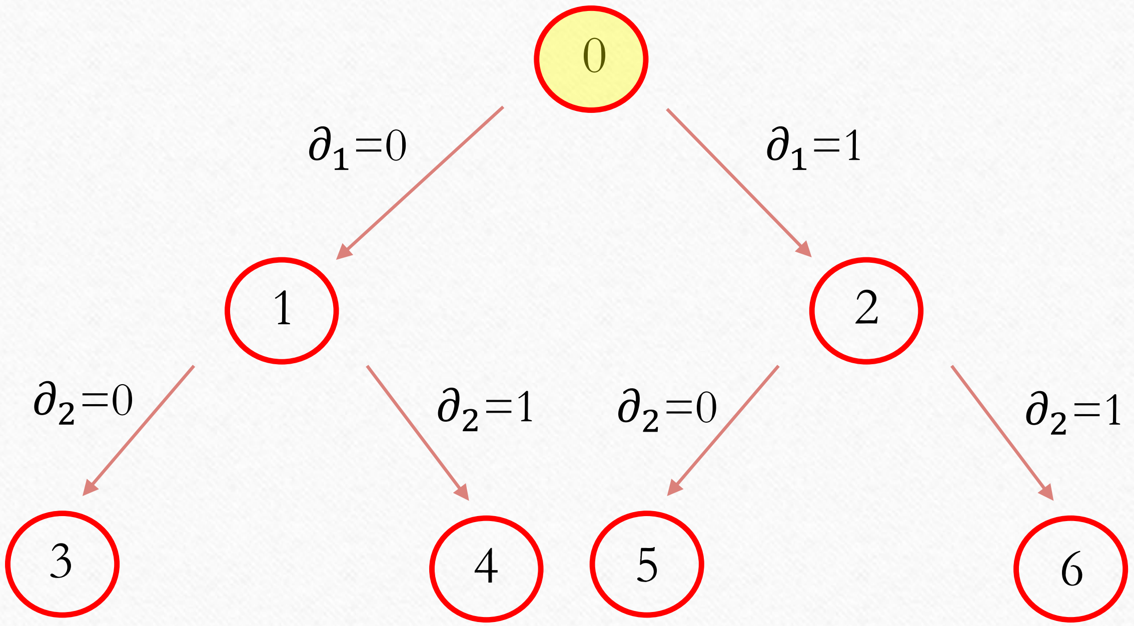


0
∂_1=0
∂_1=1
1
2
∂_2=0
∂_2=1
∂_2=0
∂_2=1
3
4
5
6

Exercise 1

Node 0 - Root: relaxed problem

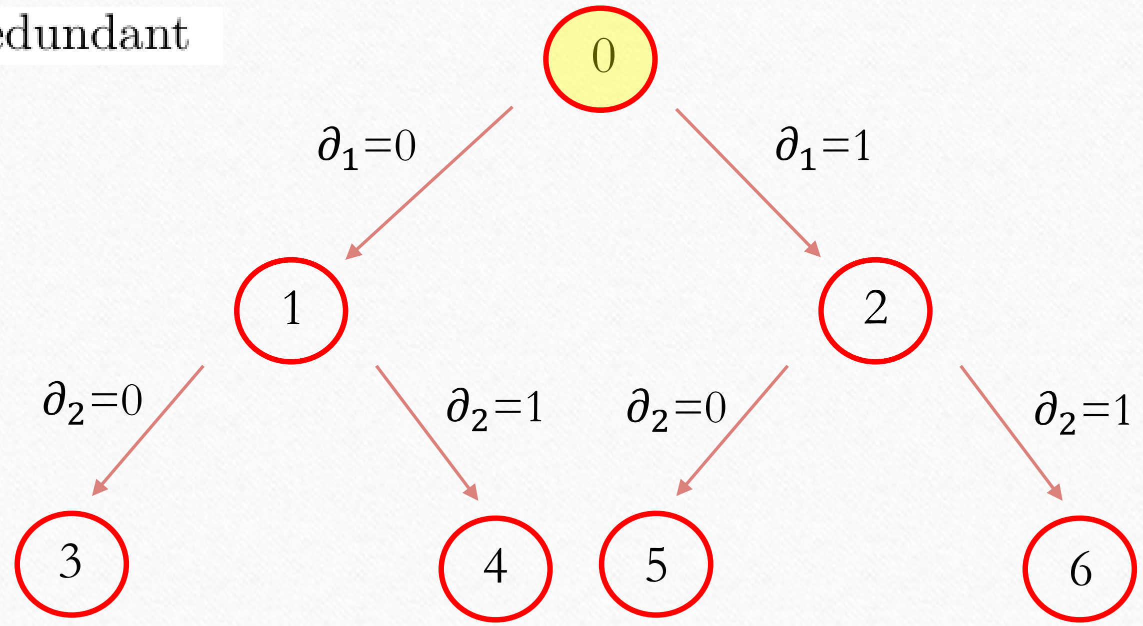
$$
\begin{aligned}
\min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 - s_1 = 100 \\
& y_1 + y_2 + s_2 = 1 \\
& y_1 + s_3 = 1 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_2, s_3, s_4, y_1, y_2 \geq 0
\end{aligned}
$$



Exercise 1

In this case the eqns. with s_3 and s_4 are redundant

$$
\begin{aligned}
\min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 - s_1 = 100 \\
& y_1 + y_2 + s_2 = 1 \\
& x_1, x_2, s_1, s_2, y_1, y_2 \geq 0
\end{aligned}
$$



Exercise 1

$$\begin{aligned}\min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\ & x_1 + 10x_2 + 2y_1 + y_2 - s_1 = 100 \\ & y_1 + y_2 + s_2 = 1 \\ & x_1, x_2, s_1, s_2, y_1, y_2 \geq 0\end{aligned}$$

$$x^\intercal = (x_1, x_2, s_1, s_2, y_1, y_2)$$

$$c^\intercal = (7, 1, 0, 0, 3, 6)$$

$$A = \begin{pmatrix} 1 & 10 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$b^\intercal = (100, 1)$$

$$\begin{aligned}\min \quad & c^\intercal x \\ & Ax = b \\ & x \geq 0\end{aligned}$$

Matlab implementation of LP:

```
opts = optimoptions(@linprog,'Algorithm','simplex');
% Use linprog with simplex algorithm
[X,FVAL,EXITFLAG,OUTPUT]=linprog(f,[],[],A,b,LB,[],[],opts)
% LB is the lower bound and contains all zeros
```

Exercise 1

$$\begin{aligned} \min \quad & 7x_1 + x_2 + 3y_1 + 6y_2 \\ & x_1 + 10x_2 + 2y_1 + y_2 - s_1 = 100 \\ & y_1 + y_2 + s_2 = 1 \\ & x_1, x_2, s_1, s_2, y_1, y_2 \geq 0 \end{aligned}$$

$$x^\intercal = (x_1, x_2, s_1, s_2, y_1, y_2)$$

$$c^\intercal = (7, 1, 0, 0, 3, 6)$$

$$A = \begin{pmatrix} 1 & 10 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$b^\intercal = (100, 1)$$

The solution to the LP relaxation of node 0 provides $x^\intercal = (0, 10, 0, 1, 0, 0)$ and cost 10. The optimizer provides binary values for y_1 and y_2 which means the optimal solution of the relaxed problem is also the optimal solution of the original MILP! (we are done!) (Yesterday we got some approximation cause we were not imposing the simplex algorithm as method)

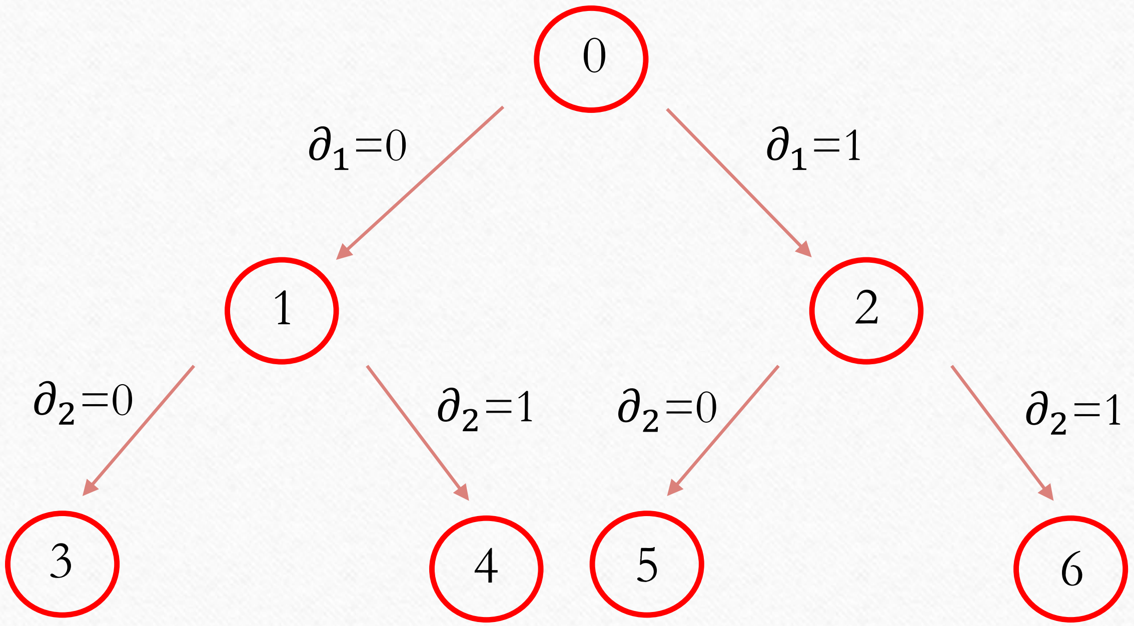
Exercise 2

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 \leq 100 \\
& y_1 + y_2 \leq 1.5 \\
& x_1, x_2 \geq 0 \\
& y_1, y_2 \in \{0, 1\}
\end{aligned}
$$

standard form

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 + s_1 = 100 \\
& y_1 + y_2 + s_2 = 1.5 \\
& x_1, x_2, s_1, s_2 \geq 0 \\
& y_1, y_2 \in \{0, 1\}
\end{aligned}
$$

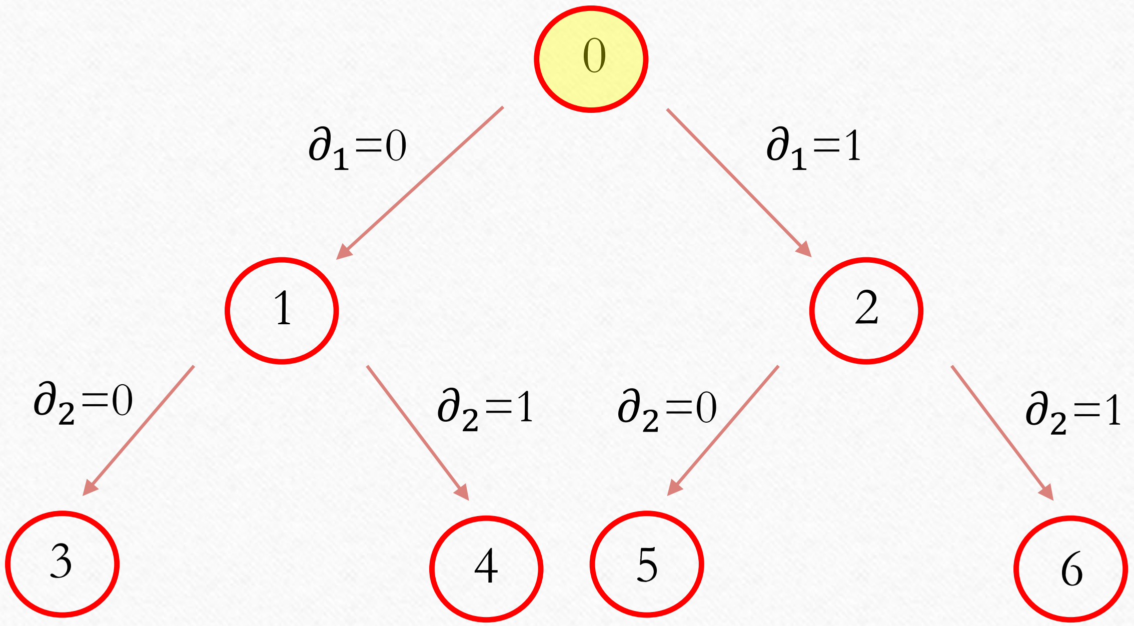
Exercise 2



Exercise 2

Node 0 - Root: relaxed problem

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 + s_1 = 100 \\
& y_1 + y_2 + s_2 = 1.5 \\
& y_1 + s_3 = 1 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_2, s_3, s_4, y_1, y_2 \geq 0
\end{aligned}
$$



Exercise 2

Node 0 - Root: relaxed problem

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 3y_1 + 6y_2 \\
& x_1 + 10x_2 + 2y_1 + y_2 + s_1 = 100 \\
& y_1 + y_2 + s_2 = 1.5 \\
& y_1 + s_3 = 1 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_2, s_3, s_4, y_1, y_2 \geq 0
\end{aligned}
$$

This time the 2 new constraints are not redundant.
Solution of the relaxed problem:

$x^\intercal = (98, 0, 0, 0, 0.5, 0, 0.5, 1)$

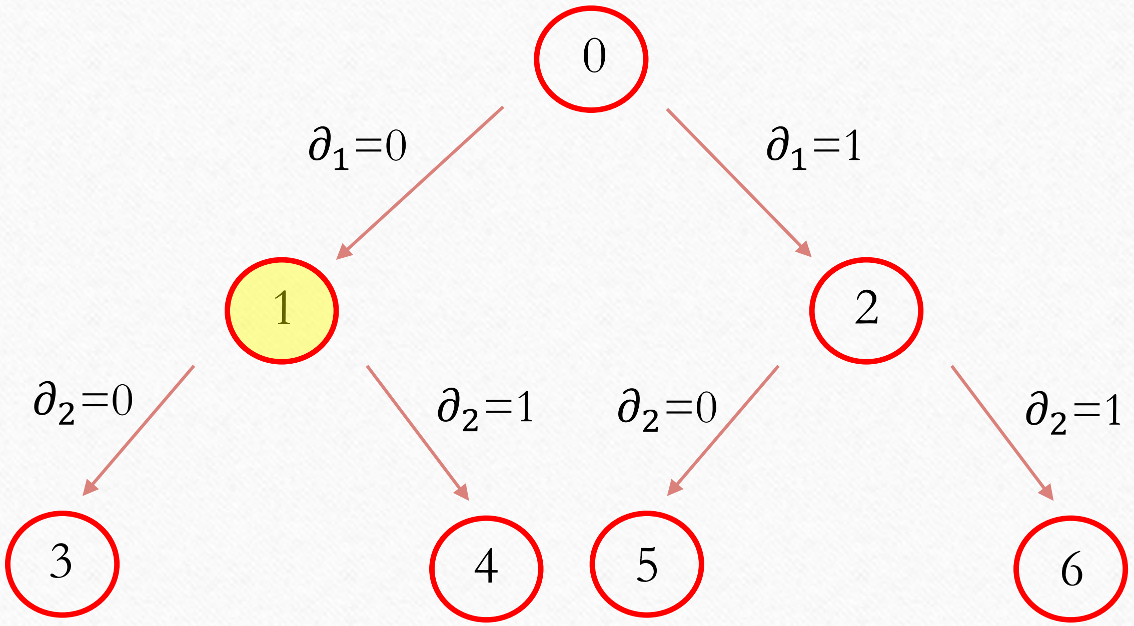
Optimal cost of the relaxed problem: 105.5

Since all the relaxed binary variables are not 0 or 1 at the optimum, the LP relaxation at the root node provides only an upper bound on the optimal cost of the original MILP (is a maximization $\rightarrow$ upper bound)

Exercise 2

Node 1 - $y_1 = 0$ (y_2 free)

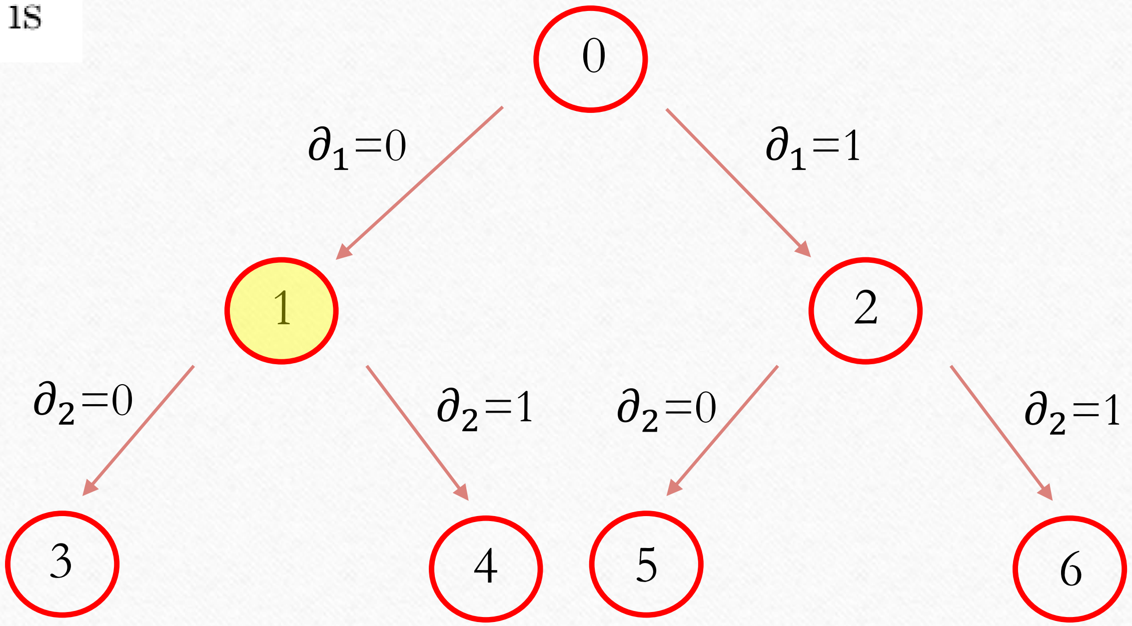
$$
\begin{aligned}
\max \quad & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 100 \\
& y_2 + s_2 = 1.5 \\
& x_1, x_2, s_1, s_2 \geq 0 \\
& y_2 \in \{0, 1\}
\end{aligned}
$$



Exercise 2

The LP relaxation of the candidate problem is

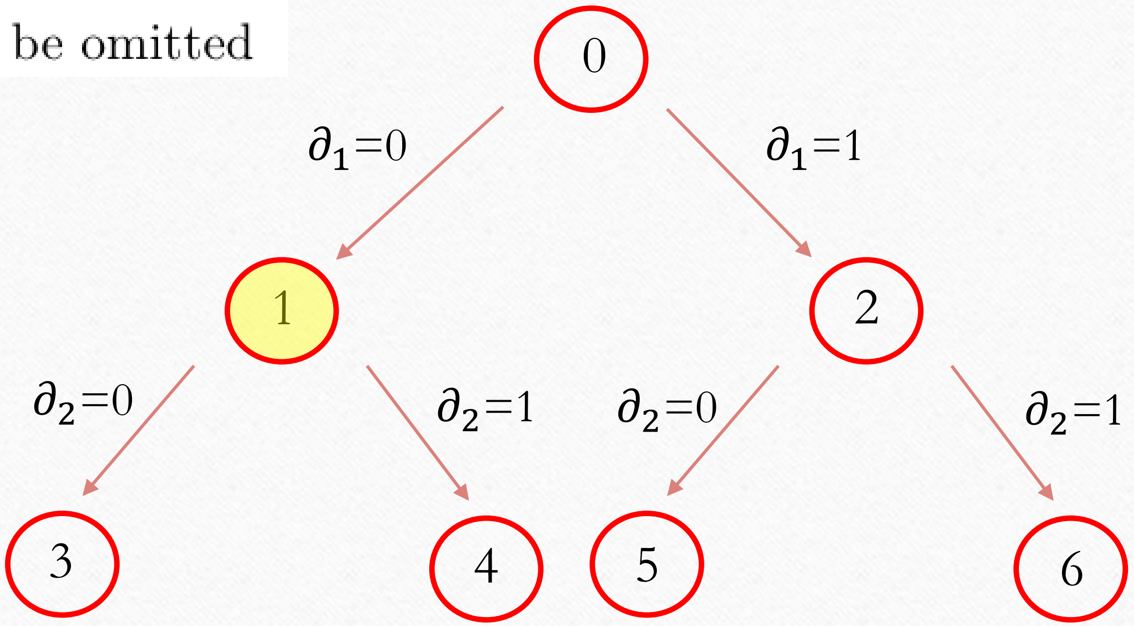
$$
\begin{aligned}
\max \quad & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 100 \\
& y_2 + s_2 = 1.5 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_2, s_4, y_2 \geq 0
\end{aligned}
$$



Exercise 2

The equation with s_2 is redundant and can be omitted

$$\begin{aligned} \max \quad & x_1 + x_2 + 6y_2 \\ & x_1 + 10x_2 + y_2 + s_1 = 100 \\ & y_2 + s_4 = 1 \\ & x_1, x_2, s_1, s_4, y_2 \geq 0 \end{aligned}$$



Exercise 2

The equation with s_2 is redundant and can be omitted

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 100 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_4, y_2 \geq 0
\end{aligned}
$$

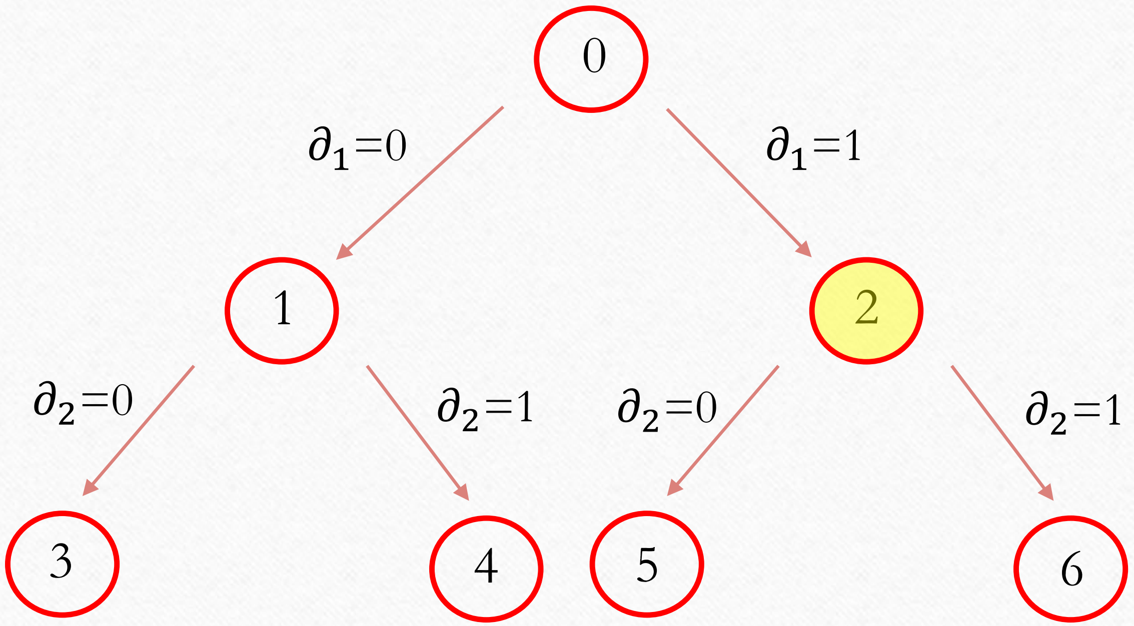
The solution to the LP relaxation of the candidate problem associated to node 1 provides $x^\intercal = (99, 0, 0, 0, 1)$ and cost 105.

This provides an **incumbent solution** since $y_1 = 0$ and $y_2 = 1$!!

Exercise 2

Node 2 - $y_1 = 1$ (y_2 free)

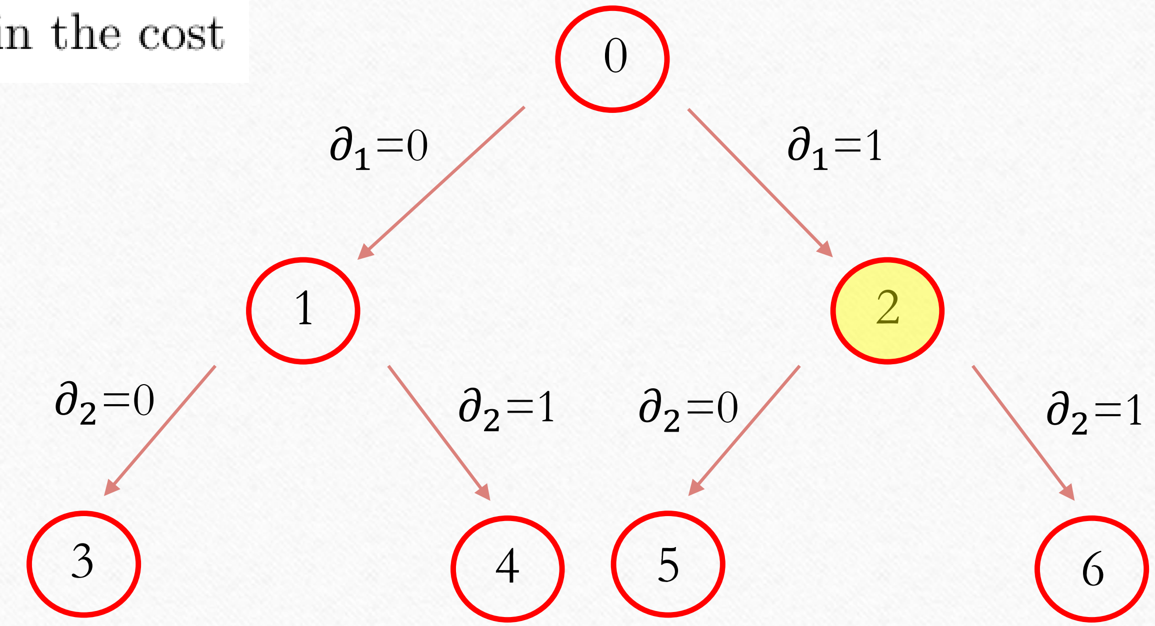
$$\begin{aligned} \max \quad & x_1 + x_2 + 3 + 6y_2 \\ & x_1 + 10x_2 + 2 + y_2 + s_1 = 100 \\ & 1 + y_2 + s_2 = 1.5 \\ & x_1, x_2, s_1, s_2 \geq 0 \\ & y_2 \in \{0, 1\} \end{aligned}$$



Exercise 2

By rearranging and dropping the constant in the cost

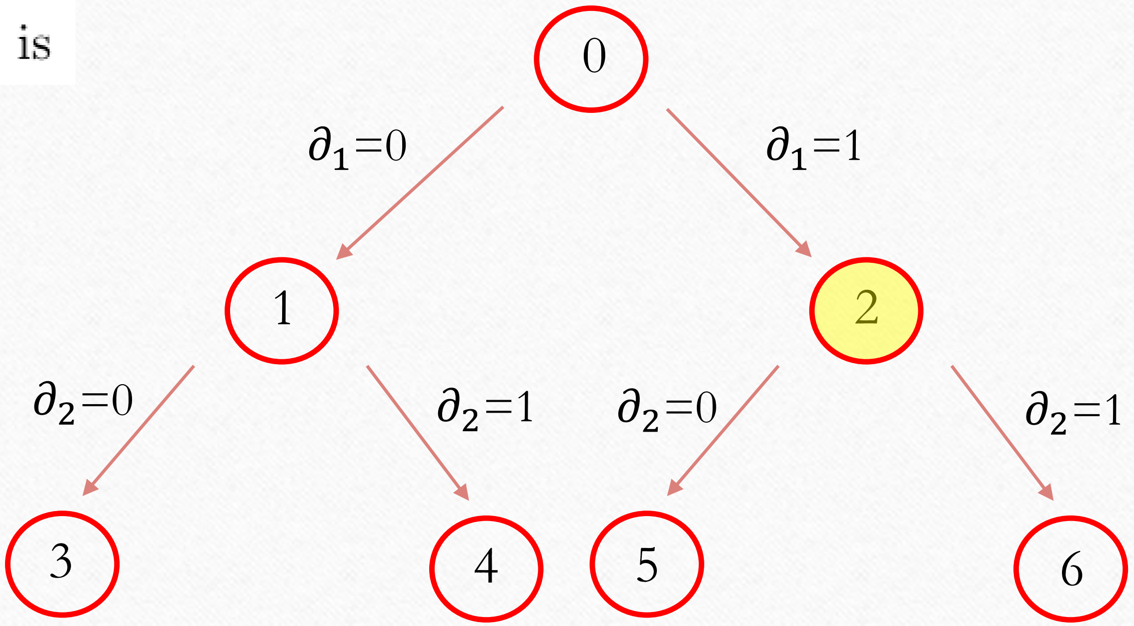
$$
\begin{array}{ll}
\max & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 98 \\
& y_2 + s_2 = 0.5 \\
& x_1, x_2, s_1, s_2 \geq 0 \\
& y_2 \in \{0, 1\}
\end{array}
$$



Exercise 2

The LP relaxation of the candidate problem is

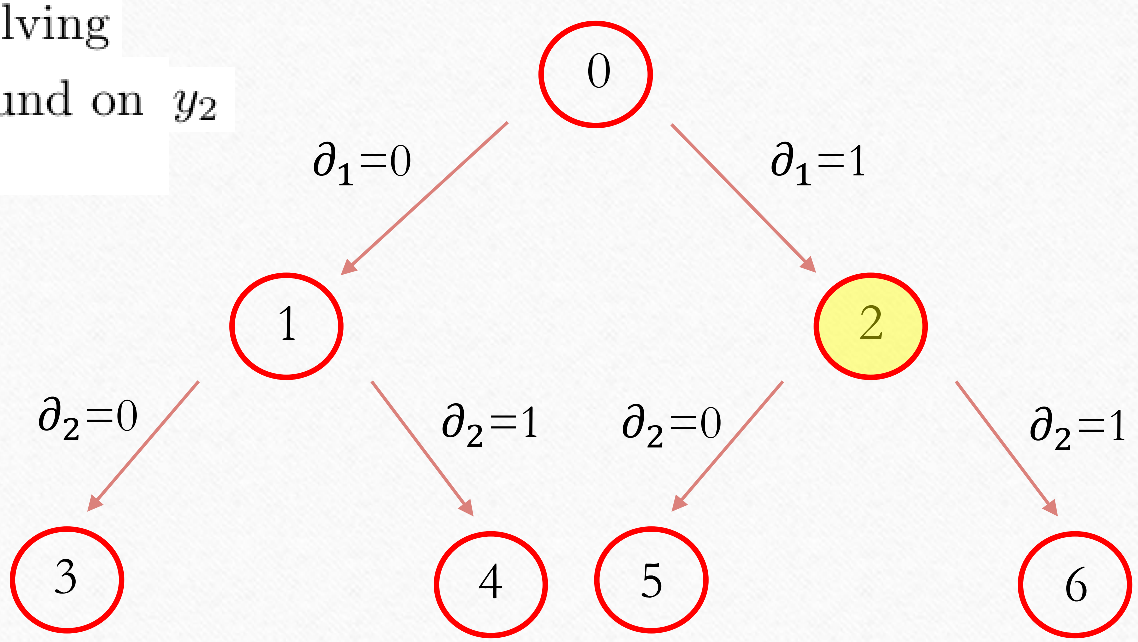
$$
\begin{aligned}
\max \quad & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 98 \\
& y_2 + s_2 = 0.5 \\
& y_2 + s_4 = 1 \\
& x_1, x_2, s_1, s_2, s_4, y_2 \geq 0
\end{aligned}
$$



Exercise 2

In this case we can drop the constraint involving s_4 since the previous one gives a tighter bound on y_2

$$\begin{array}{ll} \max & x_1 + x_2 + 6y_2 \\ & x_1 + 10x_2 + y_2 + s_1 = 98 \\ & y_2 + s_2 = 0.5 \\ & x_1, x_2, s_1, s_2, y_2 \geq 0 \end{array}$$



Exercise 2

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 6y_2 \\
& x_1 + 10x_2 + y_2 + s_1 = 98 \\
& y_2 + s_2 = 0.5 \\
& x_1, x_2, s_1, s_2, y_2 \geq 0
\end{aligned}
$$

The solution to the LP relaxation of the candidate problem associated to node 2 provides $x^\intercal = (97.5, 0, 0, 0, 0.5)$ and cost 103.5 (was 100.5+3 that we dropped).

This provides an upper bound to any completion of (1,#).

Since such upper bound is lower than the incumbent solution we have, there is no reason to explore this branch any further!

We are done.

The optimal solution is $x_1 = 99, x_2 = 0, s_1 = 0, s_2 = 0.5, y_1 = 0, y_2 = 1$ and the optimal cost is 105.

The solution was obtained solving 3 LPs rather than all the combinations (which results in 4 LPs).